EXACT SOLUTION OF THE NAVIER-STOKES EQUATIONS FOR A COMPRESSIBLE GAS

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The cases when an exact solution for the Navier-Stokes equation can be obtained, are of particular interest in investigations of the viscous fluid flows.

Few such solutions are known for a viscous, heat-conducting gas, e.g. the solution obtained in [1] for the case of a viscous gas flow in a conical nozzle when the heat transfer at the wall is governed by a special law. It is also shown that such self-similar flows are absent in the case of a channel with plane walls.

In the present paper we show that in this case we can (under certain well defined conditions) reduce the Navier-Stokes equations to a system of ordinary differential equations whose solution can be obtained either in a closed form (provided that certain viscosity laws are observed) or numerically.

1. Hamel's solution for a flow of a viscous, incompressible fluid between two mutually inclined plane walls, represents one of the exact solutions of the hydrodynamic equations.

Assuming that its analog exists for the case of a viscous heat-conducting gas, we shall seek a solution of the Navier-Stokes equations written in polar r, φ-coordinates, in the following form

$$v = v(\varphi), \qquad w = 0, \qquad h = h(\varphi)$$

where v and w are the radial and tangential velocity components and h is enthalpy.

Equations of energy, impulse and continuity and the equations of state are (we consider the case of a perfect gas)

$$\begin{gathered}
(r^\circ)^2 R \frac{\partial p^\circ}{\partial r^\circ} = \frac{\partial}{\partial \varphi}\left(\mu^\circ \frac{\partial v}{\partial \varphi}\right) - \frac{4}{3}\mu^\circ v^\circ \\
r^\circ R \frac{\partial p^\circ}{\partial \varphi} = \mu^\circ \frac{\partial v^\circ}{\partial \varphi} + \frac{4}{3}\frac{\partial}{\partial \varphi}(\mu^\circ v^\circ) \\
\frac{\partial}{\partial \varphi}\left[\frac{\mu^\circ}{\sigma}\frac{\partial h^\circ}{\partial \varphi} + (\varkappa - 1) M_1^2 \mu^\circ v^\circ \frac{\partial v^\circ}{\partial \varphi}\right] = 0 \\
\frac{\partial}{\partial r}(r^\circ \rho^\circ v^\circ) = 0, \qquad \varkappa M_1^2 p^\circ = \rho^\circ h^\circ
\end{gathered} \tag{1.1}$$

Here the dimensionless quantities $r^\circ, v^\circ, p^\circ, \rho^\circ, h^\circ$ and μ° are defined by

$$r^\circ = \frac{r}{r_1}, \quad v^\circ = \frac{v}{v_1}, \quad p^\circ = \frac{p}{\rho_1 v_1^2}, \quad \rho^\circ = \frac{\rho}{\rho_1}$$
$$h^\circ = \frac{h}{h_1}, \quad \mu^\circ = \frac{\mu}{\mu_1}, \quad R = \frac{\rho_1 v_1 r_1}{\mu_1}$$

where p is the pressure, ρ is the density, μ is the coefficient of viscosity, $\varkappa$ is the ratio of specific heats, σ is Prandtl number, M is Mach number; v_1, ρ_1, h_1, μ_1 and M_1 denote the values of the respective quantities at the distance r_1 from the coordinate origin with φ arbitrarity fixed (e.g. $\varphi = 0$) and R is the Reynolds number which is constant.

We shall investigate the power dependence of the coefficient of viscosity on temperature $\mu \sim h^n$, although, as we shall see below, self-similar solutions exist for an arbitrary relationship $\mu(h)$.

It can easily be shown that the system (1.1) reduces to a system of ordinary differential equations (from now on we shall omit the superscript ° accompanying the dimensionless quantities).

Equation of continuity yields $r\rho v = \Phi(\varphi)$ which, together with the equation of state, gives

$$p = \frac{1}{\varkappa M_1^2} p_1 p_2 \qquad \left(p_1 = p_1(r) = \frac{1}{r}, \quad p_2 = p_2(\varphi) = \Phi\frac{h}{v}\right) \tag{1.2}$$

Further, inserting (1.2) into the first two equations of (1.1) and integrating the energy equation once with respect to φ, we obtain

$$\frac{d}{d\varphi}\left(\mu\frac{dv}{d\varphi}\right) = \frac{4}{3}\mu v - \frac{R}{\varkappa M_1^2} p_2, \qquad \frac{R}{\varkappa M_1^2}\frac{dp_2}{d\varphi} = \mu\frac{dv}{d\varphi} + \frac{4}{3}\frac{d}{d\varphi}(\mu v),$$
$$\frac{\mu}{\sigma}\frac{dh}{d\varphi} + \mu\frac{\varkappa - 1}{2} M_1^2 \frac{d}{d\varphi}(v^2) = C \tag{1.3}$$

The constant of integration C characterizes the magnitude of the heat flux across the channel walls.

Since the heat flux across each wall is equal to its counterpart in magnitude and opposite in sign, therefore the heat content is constant along the streamlines although the flow need not be symmetric.

Assuming that the value $\varphi = 0$ corresponds to the condition $dv / d\varphi = 0$, we can regard the system of ordinary differential equations in v, h and p_2 as a system with initial conditions $v = 1$, $dv / d\varphi = 0$, $h = 1$, and $p_2 = 1$ when $\varphi = 0$. The corresponding angles of inclination of the channel walls to the plane $\varphi = 0$ in the positive and negative direction are obtained, after integrating Eqs. (1.3) (with given R, M_1 and C) from the condition of zero velocity on both walls.

System (1.3) with given initial conditions can be solved for the coefficient of viscosity depending arbitrarily on temperature by any numerical method on a digital computer.

2. In the following we shall consider symmetric flows only (without the heat transfer through the walls of the channel) when $C = 0$.

Integrating the last equation of (1.3) we obtain, independently of the law governing the viscosity,

$$h = 1 + \sigma\frac{\varkappa - 1}{2} M^2 (1 - v^2) \tag{2.1}$$

From this it follows that the coefficient of recombination of enthalpy (temperature) at the wall is equal to σ (M denotes the Mach number along the axis of the channel).

The quantities $R / \varkappa M^2 = \alpha$, $\sigma(\varkappa - 1) M^2 / 2 = \beta$ and n - the power index in the formula for the viscosity coefficients, are used as parameters of the system under consideration.

Putting further $Rp_2 / \varkappa M^2 = P_2$, we find that $(P_2)_{\varphi=0} = \alpha$.

When $n = 0.5$ and 1, the problem has an analytic solution as for $n = 0$. Thus, integrating the second equation of (1.3) we obtain the following relationship between P_2 and v:

$$(n = 0.5) \quad P_2(v) = \alpha - \left[\frac{11}{6} + \frac{1+\beta}{2\sqrt{\beta}} \arcsin\left(\frac{\beta}{1+\beta}\right)^{1/2}\right] +$$
$$+ \left[\frac{11}{6} v\sqrt{1 + \beta(1 - v^2)} + \frac{1+\beta}{2\sqrt{\beta}} \arcsin v\left(\frac{\beta}{1+\beta}\right)^{1/2}\right] \tag{2.2}$$
$$(n = 1) \quad P_2(v) = \alpha - (7/_3 + 2/_3\beta) + 7/_3(1+\beta)v - 2/_3\beta v^3$$

When $\beta = 0$, which corresponds to $M = 0$ and to $\varkappa = 1$ or $\sigma = 0$ with M arbitrary, both expressions yield an identical result

$$P_2(v) = \alpha - {}^7/_3 (1 - v)$$

From (2.2) we see that in the general case the dependence of P_2 on v need not be monotonic and P_2 will have its minimum value when $v \neq 0$.

We can also use (2.2) to establish the smallest values which can be assumed by α for the given β.

Obviously, the condition $P_2(0) = 0$ at the wall will correspond to the minimum value of α and in this case the channel walls divergence will be at the maximum. Thus we have

$$(n = 0.5) \quad \alpha \geqslant \alpha_* = \frac{11}{6} + \frac{1+\beta}{2\sqrt{\beta}} \arcsin \left(\frac{\beta}{1+\beta}\right)^{1/2} \tag{2.3}$$
$$(n = 1) \quad \alpha \geqslant \alpha_* = {}^7/_3 + {}^2/_3\, \beta$$

which shows that the relations between the minimum value of α_* and β are different for $n = 0.5$ and 1 (for large β the corresponding relations are proportional to $\sqrt{\beta}$ and β).

It remains to integrate the first equation of (1.3) which (taking into account (2.1) and (2.2)) can be written as

$$v'' - W(v)\,(v')^2 = Q(v) \tag{2.4}$$

where the prime denotes the differentiation with respect to φ and where the following notations are introduced:

$$W(v) = 2\beta n \frac{v}{1 + \beta(1 - v^2)}, \qquad Q(v) = \frac{4}{3}\, v - \frac{P_2(v)}{[1 + \beta(1 - v^2)]^n}$$

Performing the following variable substitution $v' = t(v)$ in (2.4) (see e.g. [3]), we obtain

$$z' + \Pi(v)\, z = Q(v) \qquad (z = {}^1/_2\, t^2,\ \Pi(v) = -2W(v)) \tag{2.5}$$

where the prime denotes differentiation with respect to v.

Then

$$z = e^{-V(v)} \left[\int Q(v)\, e^{V(v)}\, dv + C_1\right] \qquad \left(V(v) = \int \Pi(v)\, dv\right) \tag{2.6}$$

will be the general solution of (2.5).

Solution of (2.4) can be written as

$$\varphi = \pm \int \frac{dv}{\sqrt{2z}} + C_2 \tag{2.7}$$

Integration constants C_1 and C_2 should be obtained from the conditions $z = 0$ and $\varphi = 0$ when $v = 1$.

It remains to compute the integrals giving explicit dependance of the radicand in (2.7) on v and then to obtain the dependance of $\varphi(v)$. We shall omit the detailed procedure and quote the final results. Thus for $n = 0.5$ we have

$$\varphi = \arc\cos \frac{I_1(v)}{I_1(1)} \tag{2.8}$$
$$\left(I_1(v) = v/2\sqrt{1+\beta(1-v^2)} + \frac{1+\beta}{2\sqrt{\beta}} \arcsin v \left(\frac{\beta}{1+\beta}\right)^{1/2} + \Delta\alpha,\ \Delta\alpha = \alpha - \alpha_*\right)$$

from which it follows that for any β the maximum half-angle of divergence of the channel walls φ_w is obtained and is equal to $^1/_2\,\pi$ at $\Delta\alpha = 0$ $(\alpha = \alpha_*)$.

We can obtain a number of the limiting expressions derived from (2.8). For example, when $\beta = 0$ $(M = 0$, or $\varkappa = 1$, or $\sigma = 0)$, we have

$$v = (1 + \Delta\alpha)\cos\varphi - \Delta\alpha \tag{2.9}$$

The case $\beta \to \infty$ corresponds to $M \to \infty$ or $\sigma \to \infty$. Assuming in addition that $\Delta\alpha = 0$, we obtain

$$\varphi = \arc\cos\left[\frac{2}{\pi}\left(v\sqrt{1-v^2} + \arcsin v\right)\right] \tag{2.10}$$

Similarly, for $n = 1$ we have

$$\varphi = \arccos\frac{I_2(v)}{I_2(1)} \qquad (I_2(v) = (1+\beta)v - {}^1/_3\beta v^3 + \Delta\alpha) \tag{2.11}$$

from which it follows that for any value of β the maximum half-angle of divergence of the channel walls is obtained for $\Delta\alpha = 0$, and that it is again equal to $^1/_2\,\pi$. The limiting expression resulting from (2.11) for $\beta = 0$ and $\Delta\alpha$ is assumed finite, coincides with (2.9), becoming however

$$\varphi = \arc\cos\,({}^3/_2\,v - {}^1/_2\,v^3) \tag{2.12}$$

when $\beta \to \infty$ and $\Delta\alpha = 0$.

When $n = 1$, the formula (2.12) assumes a distinct feature, namely (see e.g. [3]) the quantity $(dv/d\varphi)_{\varphi=0}$ is not equal to zero, but to $-1/\sqrt{3}$. This is explained by the fact that two limit values $n = 1$ and $\beta \to \infty$ affect each other.

Next we shall investigate the behavior of the solutions (2.8) and (2.11) obtained for the case when the angle of divergence of the channel walls is small $(\varphi_w \ll 1)$. Taking into account the smallness of φ_w we have from (2.11)

$$1 - \frac{\varphi_w^2}{2} = \frac{\alpha - \alpha_*}{\alpha - {}^4/_3} \tag{2.13}$$

which shows clearly that the present case can occur when $\alpha \gg \alpha_*$. This means that sufficiently high values of R are necessary for the self-similar flows with moderate or large values of M to exist in such channels.

Rearranging (2.13) and neglecting the quantity $^4/_3$ as compared with α, we obtain

$$R\varphi_w^2 = 2\,(1 + {}^2/_3\,\beta)\,\varkappa M^2 \tag{2.14}$$

Using (2.8) and (2.11) we can obtain the relationship between the velocity and $\eta = \varphi / \varphi_w$.

$$(n=0.5) \quad \eta^2 = 1 - \frac{J_1(v)}{J_1(1)}$$

$$\left(J_1(v) = v_* \sqrt{1+\beta(1-v^2)} + \frac{1+\beta}{\sqrt{\beta}} \arcsin v \left(\frac{\beta}{1+\beta}\right)^{1/2}\right)$$

$$(n=1) \quad \eta^2 = 1 - \frac{J_2(v)}{J_2(1)} \quad (J_2(v) = (1+\beta)\, v - {}^1/_3 \beta v^3) \tag{2.15}$$

Thus we see from (2.14) and (2.15) that when the thermal and physical properties $(\varkappa, \sigma, n)$ of the gas flow are given and the values of the parameter $\alpha = R / \varkappa M^2$ are large, then all gasdynamic quantities have identical profiles with respect to the reduced coordinate $\eta = \varphi / \varphi_w$ in the channels possessing varying angles of divergence, provided that the quantity $R\varphi_w^2 = f(\varkappa, \sigma, n, M)$ is kept constant.

For $M = 0$ $(\beta = 0)$, we obtain from (2.14) and (2.15) a parabolic velocity profile and $\varphi_w = 0$ (R can be arbitrary), i.e. the Poiseuille flow in a channel of constant cross section.

For $n = 1$ and $\beta \to \infty$, formulas (2.15) and (2.12) yield $(dv / d\eta)_{\eta=0} - \sqrt{2/3}$.

For small values of $\varphi_w (\alpha \gg \alpha_*)$, taking into account (2.2), we find that the reduced transverse pressure p/p_0 (p_0 denotes the pressure at the axis) is

$$\frac{p}{p_0} = 1 + O\left(\frac{1}{\alpha}\right) \tag{2.16}$$

i.e. the pressure across the channel can be assumed constant to within the terms of the order of $(1/\alpha)$.

Considering flows in the channels with small angles of divergence we find, in particular, that the approximate boundary layer equations [3] used to describe the viscous gas flows in such channels, represent the limiting forms of the Navier-Stokes equations obtained when $\varphi_w \to 0$.

Moreover, expressions obtained in [3] for computing the transverse velocity profiles, coincide with (2.15) given here.

Figs. 1 to 5 show some results of computation. Thus, Fig. 1 and 2 give the velocity and pressure profiles in the transverse section of the channel for $\alpha = \alpha^*$ and Fig. 3 and 4 for $\alpha = 100$ at $\beta = 0, 1, 10$, relative to the quantity φ / φ_w.

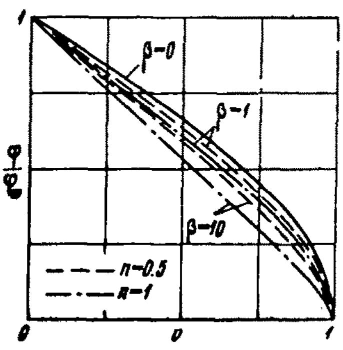

Fig. 1

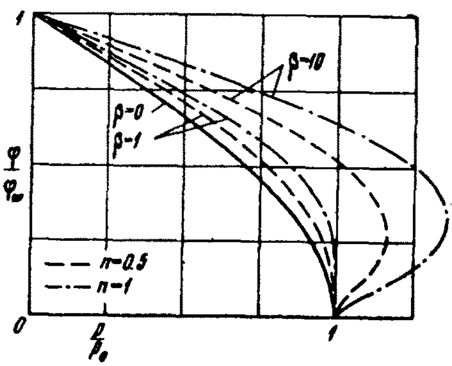

Fig. 2

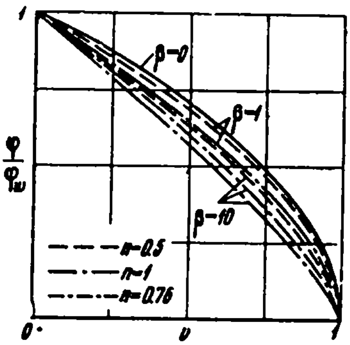

Fig. 3

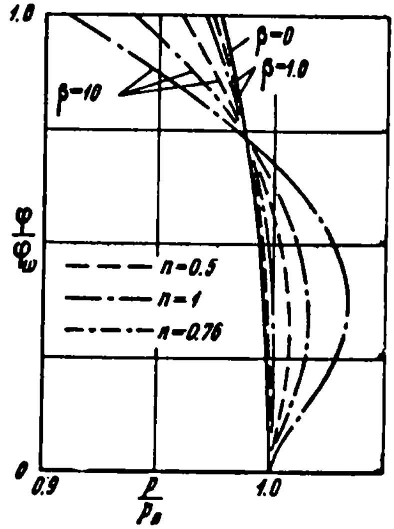

Fig. 4

Influence of the power index n in the viscosity law on the reduced characteristic curves is shown clearly. In addition, computed curves corresponding to $n = 0.76$ (air) which were obtained by numerical integration, are given for $\alpha = 100$ and $\beta = 10$.

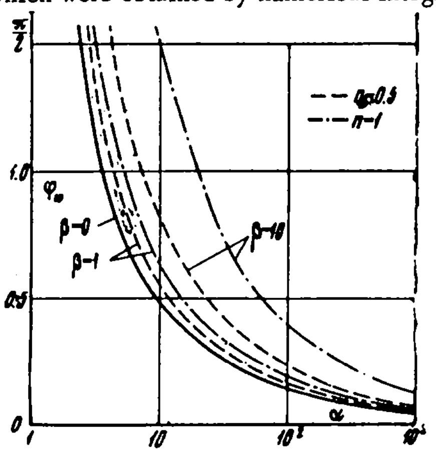

Fig. 5

Fig. 5 shows the relation between the half-angle of divergence of the channel walls φ_w and α for fixed β.

We note that the solution of the problem of the flow of a viscous, heat conducting gas in a channel can be generalized to the case of the rarefied gas flow with slip.

Since in the given case we vary only the boundary conditions at the wall, the structure of the solution will remain unchanged, but the angle of divergence of the channel walls will be different for the given values of α and β.

Moreover, $\alpha \sim 1 / \mathrm{Kn}\, M$ where $\mathrm{Kn} = l_1 / r_1$ is the Knudsen number and l_1 is the mean free path of a molecule, defined in terms of the axial flow parameters at the distance r_1.

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